Actionable information in seconds

ISCWSA Error Model Sub-Committee: XCLA Term Mathematical Derivation

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The purpose here is to define the Rev. 5 course length error as proportional to the amount of angle change in the two directions perpendicular to the wellbore at a given survey station. One of these directions is horizontal, and denoted XCLL (or XCLA for the equivalent mapping to an azimuth error) and the other is along the projection of the vertical into the plane perpendicular to the direction of wellbore motion.

First let's define some terms. With inclination denoted I and azimuth denoted A, we can define the unit vector in the direction of the wellbore as

$$\boldsymbol{u}_z^{NED} = \begin{bmatrix} \sin I \cos A \\ \sin I \sin A \\ \cos I \end{bmatrix} \tag{1}$$

where the subscript z denotes the axis of the right-handed coordinate system oriented downhole, and the superscript NED denotes that the vector is represented in the North-East-Down right-handed frame, where x points North, y East, and z Down. We also need to define the transformation matrix that transforms vectors from representations in the NED frame to the High Side (HS) frame, which has the positive z axis oriented downhole, the y axis "up" in the plane perpendicular to the wellbore, and the x axis pointing left in said plane:

$$\begin{aligned} \boldsymbol{T}_{NED}^{HS} &= \boldsymbol{R}_1(-I)\boldsymbol{R}_3(A-90^\circ) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{bmatrix} \begin{bmatrix} \sin A & -\cos A & 0 \\ \cos A & \sin A & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} sA & -cA & 0 \\ cIcA & cIsA & -sI \\ sIcA & sIsA & cI \end{bmatrix} \end{aligned} \tag{2}$$

where the abbreviations $s() = \sin()$ and $c() = \cos()$ have been used for brevity.

Now define the small angles ϕ and θ to be the angles about which the x and y axes of the HS frame are rotated to go from one station to the next. (Note that we should really use negative angles here since the vector is being rotated rather than the frame; however we only care about the magnitude of the angle change in this exercise defined in terms of the respective inclinations and azimuths, so we will use positive values to avoid carrying around extra negative signs.) Since we know I and A at stations 1 and 2, we can find the angles by solving the following equation:

$$\left(\boldsymbol{u}_{z_2}^{NED}\right)^T = \left(\boldsymbol{u}_z^{HS}\right)^T \boldsymbol{T}_{HS_1}^{HS_2} \boldsymbol{T}_{NED}^{HS_1} \tag{3}$$

where it is to be noted that $\left(\boldsymbol{u}_z^{HS}\right)^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, and

$$\begin{aligned} \boldsymbol{T}_{HS_1}^{HS_2} &= \boldsymbol{R}_1(\phi)\boldsymbol{R}_2(\theta) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix} \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \\ &= \begin{bmatrix} c\theta & 0 & -s\theta \\ s\phi s\theta & c\phi & s\phi c\theta \\ c\phi s\theta & -s\phi & c\phi c\theta \end{bmatrix} \end{aligned} \tag{4}$$

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Substituting the above into Equation 3 gives

$$
\begin{aligned}
\left(\boldsymbol{u}_{z_2}^{NED}\right)^T &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & -s\theta \\ s\phi s\theta & c\phi & s\phi c\theta \\ c\phi s\theta & -s\phi & c\phi c\theta \end{bmatrix} \begin{bmatrix} sA_1 & -cA_1 & 0 \\ cI_1 cA_1 & cI_1 sA_1 & -sI_1 \\ sI_1 cA_1 & sI_1 sA_1 & cI_1 \end{bmatrix} \\
&= \begin{bmatrix} c\phi s\theta & -s\phi & c\phi c\theta \end{bmatrix} \begin{bmatrix} sA_1 & -cA_1 & 0 \\ cI_1 cA_1 & cI_1 sA_1 & -sI_1 \\ sI_1 cA_1 & sI_1 sA_1 & cI_1 \end{bmatrix} \\
&\approx \begin{bmatrix} \theta & -\phi & 1 \end{bmatrix} \begin{bmatrix} sA_1 & -cA_1 & 0 \\ cI_1 cA_1 & cI_1 sA_1 & -sI_1 \\ sI_1 cA_1 & sI_1 sA_1 & cI_1 \end{bmatrix} \\
&= \begin{bmatrix} (\theta sA_1 - \phi cI_1 cA_1 + sI_1 cA_1) & (-\theta cA_1 - \phi cI_1 sA_1 + sI_1 sA_1) & (\phi sI_1 + cI_1) \end{bmatrix} \qquad (5)
\end{aligned}
$$

where the small angle assumption has been used for angles ϕ and θ.

Combining Equations 1 and 5 gives three equations in 2 unknowns. First, let's solve the 3^{rd} equation for ϕ:

$$
\begin{aligned}
\cos I_2 &= \phi \sin I_1 + \cos I_1 \\
&\approx \cos(I_1 - \phi) \qquad (6)
\end{aligned}
$$

where the approximation can be deduced by reversing the small angle assumption for ϕ in Equation 5. This leads to the solution

$$\phi = I_1 - I_2 \qquad (7)$$

Now we can turn our attention to finding θ. Note that the 1^{st} and 2^{nd} equations from Equation 5 are

$$sI_2 cA_2 = \theta sA_1 - \phi cI_1 cA_1 + sI_1 cA_1 \qquad (8)$$

and

$$sI_2 sA_2 = -\theta cA_1 - \phi cI_1 sA_1 + sI_1 sA_1 \qquad (9)$$

To eliminate ϕ, combine the equations in the form

$$(1^{st} \text{ equation}) \cdot sA_1 - (2^{nd} \text{ equation}) \cdot cA_1 \qquad (10)$$

which gives

$$sI_2 cA_2 sA_1 - sI_2 sA_2 cA_1 = \theta s^2 A_1 + \theta c^2 A_1 = \theta \qquad (11)$$

This can be further simplified to

$$\theta = \sin I_2 \sin(A_1 - A_2) \qquad (12)$$

Since we are only concerned with the magnitude of the angle changes, we can define the angle change for each of the XCL terms as follows:

$$
\begin{aligned}
XCLH: &\quad \phi = |I_2 - I_1| \\
XCLL: &\quad \theta = |\sin(A_2 - A_1)| \sin I_2 \\
XCLA: &\quad \theta / \sin I_2 = |\sin(A_2 - A_1)|
\end{aligned}
$$

On the last equation, see for instance the partial derivative of the error model toolface dependent term MX with respect to azimuth (MX being the angle along the HS x-axis; alternatively defined, it is the rotation of the HS frame about the HS y-axis which is the same definition as θ).

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