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Correct Vertical Well Weighting Functions for Cross-Axis Accelerometer Bias Terms

January 28, 2016

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The vertical well weighting functions corresponding to x and y accelerometer biases, as used in the ISCWSA error models, are examined herein. It is found that the toolface-dependent weighting terms as found in SPE 67616 are incorrect when applied to surveys taken with random toolface angles. They do, however, work fine if the surveys have a constant toolface orientation. The toolface-independent terms as found in Copsegrove memo CDR-SM-03, Rev. 4, with the exception of the unintentional omission of a multiplier term, are correct and can be used with either constant or random toolface angles. Care should be taken, however, to make sure that the omitted multiplier has not been omitted from an error model code.

Preliminaries

In the ISCWSA error models, the partial derivative describing the effect of an error source on position is usually given by the chain rule:

$$\frac{\partial \Delta \boldsymbol{r}_k}{\partial \epsilon_i} = \frac{\partial \Delta \boldsymbol{r}_k}{\partial \boldsymbol{p}_k} \frac{\partial \boldsymbol{p}_k}{\partial \boldsymbol{\epsilon}_i} \tag{1}$$

where $\Delta \boldsymbol{r}_k$ is the change in wellbore position from survey station $k-1$ to survey station k, and $\boldsymbol{p}_k$ is a vector of derived survey measurements at survey station k such that

$$\boldsymbol{p}_k = \begin{bmatrix} D_k \\ I_k \\ A_k \end{bmatrix} \tag{2}$$

Here, D_k is along hole measured depth, I_k is calculated inclination, and A_k is calculated true azimuth.

A problem arises with the chain rule paradigm in vertical holes. In such cases, the partial derivative $\frac{\partial \boldsymbol{p}_k}{\partial \boldsymbol{\epsilon}_i}$ can become singular for some error sources. A prominent example is the partial derivative of azimuth with respect to accelerometer x and y axis biases. However, when Equation 1 is evaluated as a single entity, rather than as a chain, the singularity vanishes. The purpose of this memo is to ascertain the proper form of these vertical orientation partial derivatives.

Singular Weighting Function Substitutions for Toolface-Dependent Error Models

For simplicity, and in keeping with [Williamson(2000)], the balanced tangential method will be used for position propagation:

$$\Delta \boldsymbol{r}_k = \frac{D_k - D_{k-1}}{2} \begin{bmatrix} \sin I_{k-1} \cos A_{k-1} + \sin I_k \cos A_k \\ \sin I_{k-1} \sin A_{k-1} + \sin I_k \sin A_k \\ \cos I_{k-1} + \cos I_k \end{bmatrix} \tag{3}$$

Clearly, we need the form of the sines and cosines of inclination and azimuth. Define the gravitational acceleration in the sensor frame as

$$\boldsymbol{G} = \begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} \tag{4}$$

where it is assumed that the z axis points downhole, and the x and y axes are such that the coordinate system is right-handed. Then we can write

$$\begin{aligned} \sin I &= \frac{G_z}{\|\boldsymbol{G}\|} \\ \cos I &= \frac{\sqrt{G_x^2 + G_y^2}}{\|\boldsymbol{G}\|} \end{aligned} \tag{5}$$

Likewise, the magnetic field vector due to the Earth in the same sensor frame is defined as

$$\boldsymbol{B} = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} \tag{6}$$

Calculating the sine and cosine of azimuth is a little trickier. First note that a unit vector pointing East can be defined as

$$\boldsymbol{u}_E = \frac{\boldsymbol{G} \times \boldsymbol{B}}{\|\boldsymbol{G} \times \boldsymbol{B}\|} \tag{7}$$

Further, a unit vector pointing in North can be defined as

$$\boldsymbol{u}_N = \frac{\boldsymbol{u}_E \times \boldsymbol{G}}{\|\boldsymbol{G}\|} = \frac{(\boldsymbol{G} \times \boldsymbol{B}) \times \boldsymbol{G}}{\|\boldsymbol{G}\|\|\boldsymbol{G} \times \boldsymbol{B}\|} \tag{8}$$

If we define a unit vector along the z axis in the sensor frame, $\boldsymbol{u}_z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, and we note that the magnitude of the horizontal component of $\boldsymbol{u}_z$ is

$$\boldsymbol{u}_{z_h} = \sin I \tag{9}$$

then we can write the sine and cosine of azimuth as

$$\begin{aligned} \sin A &= \frac{\boldsymbol{u}_E^T \boldsymbol{u}_z}{\sin I} \\ \cos A &= \frac{\boldsymbol{u}_N^T \boldsymbol{u}_z}{\sin I} \end{aligned} \tag{10}$$

We can put these equations in a more familiar form by first noting that, from the definition of a vector cross product, we can write

$$\begin{aligned} \|\boldsymbol{G} \times \boldsymbol{B}\| &= \|\boldsymbol{G}\|\|\boldsymbol{B}\| \sin(90^\circ - \Theta) \\ &= \|\boldsymbol{G}\|\|\boldsymbol{B}\| \cos\Theta \end{aligned} \tag{11}$$

where Θ is the magnetic Dip angle. Using Equation 11, and carrying out the dot and cross products, we can get

$$\begin{aligned} \sin A &= \frac{(G_x B_y - G_y B_x)}{\|\boldsymbol{G}\|\|\boldsymbol{B}\| \cos\Theta \sin I} \\ \cos A &= \frac{B_z(G_x^2 + G_y^2) - G_z(G_x B_x + G_y B_y)}{\|\boldsymbol{G}\|^2\|\boldsymbol{B}\| \cos\Theta \sin I} \end{aligned} \tag{12}$$

In order to assess the impact of accelerometer bias on position propagation in a vertical well, let us assume that nominal inclination is zero. So $\sin I = 0$ and $\cos I = 1$ nominally. Also, since azimuth is undefined in a vertical well, it is incorrect to continue to refer to the sine and cosine of azimuth when $I = 0$. Rather, we have two calculated values that, when multiplied by $\sin I$, equal to zero since $G_x = G_y = 0$ and the sines cancel:

$$\begin{aligned} F_s &= \frac{(G_x B_y - G_y B_x)}{\|\boldsymbol{G}\|\|\boldsymbol{B}\| \cos\Theta \sin I} \\ F_c &= \frac{B_z(G_x^2 + G_y^2) - G_z(G_x B_x + G_y B_y)}{\|\boldsymbol{G}\|^2\|\boldsymbol{B}\| \cos\Theta \sin I} \end{aligned} \tag{13}$$

Now consider a small deviation in the x axis accelerometer reading, ΔG_x, such that $G_x = 0 + \Delta G_x$. Let us determine the effect of this bias on Δr_k due to survey station k. First note that the bias will cause a deviation in inclination:

$$\begin{aligned} \sin(0 + \Delta I) &= \sin \Delta I \\ \cos(0 + \Delta I) &= \cos \Delta I \end{aligned} \tag{14}$$

Next note that the calculated variables will also have a deviation due to the bias

$$\begin{aligned} F_s &= \Delta F_s \\ F_c &= \Delta F_c \end{aligned} \tag{15}$$

Putting this together in the definition of a derivative, and recalling that we are only examining the effect at survey station k, we get

$$
\begin{aligned}
\frac{\partial \Delta \boldsymbol{r}_k}{\partial \Delta G_x} &= \lim_{\Delta G_x \to 0} \frac{\Delta \boldsymbol{r}_k(\Delta G_x) - \Delta \boldsymbol{r}_k(0)}{\Delta G_x} \\
&= \lim_{\Delta G_x \to 0} \frac{D_k - D_{k-1}}{2\Delta G_x} \begin{bmatrix} \sin \Delta I_k \Delta F_{c_k} - 0 \\ \sin \Delta I_k \Delta F_{s_k} - 0 \\ \cos \Delta I_k - 1 \end{bmatrix} \\
&= \lim_{\Delta G_x \to 0} \frac{D_k - D_{k-1}}{2\Delta G_x} \begin{bmatrix} \frac{\sin \Delta I (B_z(\Delta G_x^2 + 0) - G_z(\Delta G_x B_x + 0))}{\|\boldsymbol{G}\|^2 \|\boldsymbol{B}\| \cos\Theta \sin \Delta I} \\ \\ \frac{\sin \Delta I (\Delta G_x B_y - 0)}{\|\boldsymbol{G}\| \|\boldsymbol{B}\| \cos\Theta \sin \Delta I} \\ \\ \frac{G_z}{\sqrt{G_z^2 + \Delta G_x^2}} - 1 \end{bmatrix} \\
&= \frac{D_k - D_{k-1}}{2} \begin{bmatrix} \frac{-B_x}{\|\boldsymbol{G}\| \|\boldsymbol{B}\| \cos\Theta} \\ \\ \frac{B_y}{\|\boldsymbol{G}\| \|\boldsymbol{B}\| \cos\Theta} \\ \\ 0 \end{bmatrix} \\
&= \frac{D_k - D_{k-1}}{2} \begin{bmatrix} \frac{-\sin \alpha_m}{\|\boldsymbol{G}\|} \\ \\ \frac{\cos \alpha_m}{\|\boldsymbol{G}\|} \\ \\ 0 \end{bmatrix}
\end{aligned} \tag{16}
$$

where $alpha_m$ is the magnetic toolface angle the subscript k has been dropped for brevity. To clarify, when vertical

$$
\begin{aligned}
B_x &= \|\boldsymbol{B}\| \cos\Theta \sin \alpha_m \\
B_y &= \|\boldsymbol{B}\| \cos\Theta \cos \alpha_m
\end{aligned} \tag{17}
$$

A similar analysis for y axis accelerometer bias yields

$$
\begin{aligned}
\frac{\partial \Delta \boldsymbol{r}_k}{\partial \Delta G_y} &= \lim_{\Delta G_y \to 0} \frac{\Delta \boldsymbol{r}_k(\Delta G_y) - \Delta \boldsymbol{r}_k(0)}{\Delta G_y} \\
&= \lim_{\Delta G_y \to 0} \frac{D_k - D_{k-1}}{2\Delta G_y} \begin{bmatrix} \sin \Delta I_k \Delta F_{c_k} - 0 \\ \sin \Delta I_k \Delta F_{s_k} - 0 \\ \cos \Delta I_k - 1 \end{bmatrix} \\
&= \lim_{\Delta G_y \to 0} \frac{D_k - D_{k-1}}{2\Delta G_y} \begin{bmatrix} \frac{\sin \Delta I (B_z(0+\Delta G_y^2) - G_z(0+\Delta G_y B_y))}{\|\boldsymbol{G}\|^2 \|\boldsymbol{B}\| \cos\Theta \sin \Delta I} \\ \\ \frac{\sin \Delta I (0 - \Delta G_y B_x)}{\|\boldsymbol{G}\| \|\boldsymbol{B}\| \cos\Theta \sin \Delta I} \\ \\ \frac{G_z}{\sqrt{G_z^2 + \Delta G_y^2}} - 1 \end{bmatrix} \\
&= \frac{D_k - D_{k-1}}{2} \begin{bmatrix} \frac{-B_y}{\|\boldsymbol{G}\| \|\boldsymbol{B}\| \cos\Theta} \\ \\ \frac{-B_x}{\|\boldsymbol{G}\| \|\boldsymbol{B}\| \cos\Theta} \\ \\ 0 \end{bmatrix} \\
&= \frac{D_k - D_{k-1}}{2} \begin{bmatrix} \frac{-\cos \alpha_m}{\|\boldsymbol{G}\|} \\ \\ \frac{-\sin \alpha_m}{\|\boldsymbol{G}\|} \\ \\ 0 \end{bmatrix}
\end{aligned} \tag{18}
$$

To summarize, when in a vertical hole, the chain rule calculation used to determine the effect of x and y accelerometer biases should be replaced by

$$
\boxed{\frac{\partial \Delta \boldsymbol{r}_k}{\partial ABX} = \frac{D_k - D_{k-1}}{2} \begin{bmatrix} \frac{-\sin \alpha_m}{\|\boldsymbol{G}\|} \\ \\ \frac{\cos \alpha_m}{\|\boldsymbol{G}\|} \\ \\ 0 \end{bmatrix}} \tag{19}
$$

and

$$
\boxed{\frac{\partial \Delta \boldsymbol{r}_k}{\partial ABY} = \frac{D_k - D_{k-1}}{2} \begin{bmatrix} \frac{-\cos \alpha_m}{\|\boldsymbol{G}\|} \\ \\ \frac{-\sin \alpha_m}{\|\boldsymbol{G}\|} \\ \\ 0 \end{bmatrix}} \tag{20}
$$

Comparison with the Williamson Paper Formulation

SPE 67616 has formulations for accelerometer x and y bias weighting functions in vertical holes. If one assumes that the computer code in use will calculate $A = 0$ and $\alpha = 0$ when $I = 0$ (α is high-side toolface), then simulation shows that the position uncertainty resulting from the formulation is identical to that resulting from Equations 19 and 20 for sliding operations. However, the SPE 67616 formulation significantly overstates the position uncertainty for rotating operations. This can be see in Figure 1, where the green ellipse of uncertainty (EOU), calculated with Equations 19 and 20, is shown to match the blue ellipse calculated from the covariance of the monte carlo samples. However, the red ellipse, calculated from the formulation presented in SPE 67616, is shown to be significantly in error.

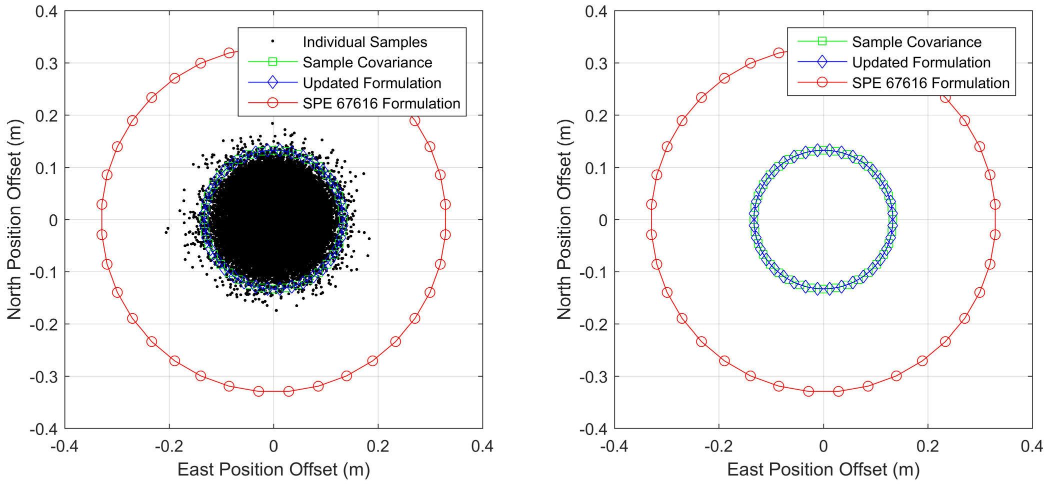

Figure 1: Comparison of Horizontal Position Uncertainty Predictions in a Vertical Well Under Rotating Conditions After 300 Meters. Plots are Identical Except the Left Plot Shows the Samples Drawn from a 50,000 Run Monte Carlo Analysis.

Singular Weighting Function Substitutions for Toolface-Independent Error Models

This section addresses the toolface-invariant fictive error sources introduced in [Torkildsen and Bang(2000)]. Table 1 in [Williamson(2000)] indicates that the partial derivatives, $\frac{\partial \boldsymbol{p}_k}{\partial \epsilon_i}$, for the x and y accelerometer bias terms are first order in $\cos \tau_k$ and $\sin \tau_k$. Thus, they only have non-zero weighting terms for fictive error sources 4 and 5 in Appendix E of [Torkildsen and Bang(2000)]. Note that the test [Torkildsen and Bang(2000)] actually gives the weighting functions as $\frac{\partial \Delta \boldsymbol{r}_k}{\partial \epsilon_i}$, where ϵ_i is a fictive error source. Thus, in order to stick with the definition of the fictive error sources, we will analyze the full derivative analytic expression directly.

Accelerometer x Axis Bias, Term 4

The 4^{th} fictive error source term corresponding to x accelerometer bias has a weighting function, for a stationary survey k, given by

$$\begin{aligned}
\sqrt{\frac{1}{2}} d_{X_k} &= \sqrt{\frac{1}{2}} \frac{\partial \Delta \boldsymbol{r}_k}{\partial \boldsymbol{p}_k} \frac{\partial \boldsymbol{p}_k}{\partial \boldsymbol{\epsilon}_i} \\
&= \frac{D_k - D_{k-1}}{2\sqrt{2}} \left[\begin{array}{c|c|c} \frac{1}{\Delta D_k} \frac{\partial \Delta \boldsymbol{r}_k}{\partial D_k} & \begin{array}{c} \cos I \cos A \\ \cos I \sin A \\ -\sin I \end{array} & \begin{array}{c} -\sin I \sin A \\ \sin I \cos A \\ 0 \end{array} \end{array} \right] \left[\begin{array}{c} 0 \\ u_k^I \\ u_k^A \end{array} \right] \\
&= \frac{\Delta D_k}{\|\boldsymbol{G}\| 2\sqrt{2}} \left[\begin{array}{c|c|c} \frac{1}{\Delta D_k} \frac{\partial \Delta \boldsymbol{r}_k}{\partial D_k} & \begin{array}{c} \cos I \cos A \\ \cos I \sin A \\ -\sin I \end{array} & \begin{array}{c} -\sin I \sin A \\ \sin I \cos A \\ 0 \end{array} \end{array} \right] \left[\begin{array}{c} 0 \\ -\cos I \\ F_s \cos I \tan \Theta \end{array} \right] \\
&= \frac{\Delta D_k}{\|\boldsymbol{G}\| 2\sqrt{2}} \left[\begin{array}{c} -F_c \cos^2(I) - F_s^2 \sin I \cos I \tan \Theta \\ -F_s \cos^2(I) + F_s F_c \sin I \cos I \tan \Theta \\ \sin I \cos I \end{array} \right]
\end{aligned} \tag{21}$$

When $I \to 0$ with $G_y = 0$ and $\Delta G_x \to 0^+$, we can find the derivative of this fictive error source with respect to x axis accelerometer bias as the limit

$$\begin{aligned}
\sqrt{\frac{1}{2}} d_{X_k} &= \lim_{\Delta G_x \to 0^+} \frac{\Delta D_k \cos \Delta I}{\|\boldsymbol{G}\| 2\sqrt{2}} \left[\begin{array}{c} -\frac{\cos \Delta I (B_z(\Delta G_x^2 + 0) - G_z(\Delta G_x B_x + 0))}{\|\boldsymbol{G}\|^2 \|\boldsymbol{B}\| \cos \Theta \sin \Delta I} - \frac{(\Delta G_x B_y - 0)^2}{\|\boldsymbol{G}\|^2 \|\boldsymbol{B}\|^2 \cos^2 \Theta \sin \Delta I} \tan \Theta \\ -\frac{\cos \Delta I (\Delta G_x B_y - 0)}{\|\boldsymbol{G}\| \|\boldsymbol{B}\| \cos \Theta \sin \Delta I} + \frac{(\Delta G_x B_y - 0)(B_z(\Delta G_x^2 + 0) - G_z(\Delta G_x B_x + 0))}{\|\boldsymbol{G}\|^3 \|\boldsymbol{B}\|^2 \cos^2 \Theta \sin \Delta I} \tan \Theta \\ \sin \Delta I \end{array} \right] \\
&= \lim_{\Delta G_x \to 0^+} \frac{\Delta D_k \cos \Delta I}{\|\boldsymbol{G}\| 2\sqrt{2}} \left[\begin{array}{c} -\frac{\cos \Delta I (B_z \Delta G_x^2 - G_z \Delta G_x B_x)}{\|\boldsymbol{G}\| \|\boldsymbol{B}\| \cos \Theta |\Delta G_x|} - \frac{(\Delta G_x B_y)^2}{\|\boldsymbol{G}\| \|\boldsymbol{B}\|^2 \cos^2 \Theta |\Delta G_x|} \tan \Theta \\ -\frac{\cos \Delta I \Delta G_x B_y}{\|\boldsymbol{B}\| \cos \Theta |\Delta G_x|} + \frac{\Delta G_x^3 B_y B_z - G_z \Delta G_x^2 B_x B_y)}{\|\boldsymbol{G}\|^2 \|\boldsymbol{B}\|^2 \cos^2 \Theta |\Delta G_x|} \tan \Theta \\ \sin \Delta I \end{array} \right] \\
&= \frac{\Delta D_k}{2\sqrt{2}} \left[\begin{array}{c} \frac{\sin \alpha_m}{\|\boldsymbol{G}\|} \\ \frac{-\cos \alpha_m}{\|\boldsymbol{G}\|} \\ 0 \end{array} \right]
\end{aligned} \tag{22}$$

So we have arrived at an answer using the limit as $\Delta G_x \to 0^+$. However, if we had used the limit as $\Delta G_x \to 0^-$ the sign of Equation 22 would be positive instead of negative. This means that there is no continuity in the limit, and therefore the limit does not exist. The weighting function for the 4^{th} fictive error term does not exist on its own. However, the limit of the square of the fictive error weighting term does exist, and that is what is used for error propagation:

$$\frac{1}{2}d^2_{X_k} = \frac{\Delta D_k}{8}\begin{bmatrix} \frac{\sin^2\alpha_m}{\|\boldsymbol{G}\|^2} \\ \\ \frac{\cos^2\alpha_m}{\|\boldsymbol{G}\|^2} \\ \\ 0 \end{bmatrix} \tag{23}$$

Accelerometer y Axis Bias, Term 4

Let us now look at the effect due to the y axis accelerometer bias. The analog of Equation 21 for the y axis bias is given by

$$\begin{aligned} \sqrt{\frac{1}{2}}d_{X_k} &= \frac{D_k - D_{k-1}}{2\sqrt{2}}\left[\begin{array}{c|c|c} \frac{1}{\Delta D_k}\frac{\partial \Delta \boldsymbol{r}_k}{\partial D_k} & \begin{matrix}\cos I\cos A\\ \cos I\sin A\\ -\sin I\end{matrix} & \begin{matrix}-\sin I\sin A\\ \sin I\cos A\\ 0\end{matrix}\end{array}\right]\begin{bmatrix}0\\ u_k^I\\ u_k^A\end{bmatrix} \\ &= \frac{\Delta D_k}{\|\boldsymbol{G}\|2\sqrt{2}}\left[\begin{array}{c|c|c} \frac{1}{\Delta D_k}\frac{\partial \Delta \boldsymbol{r}_k}{\partial D_k} & \begin{matrix}\cos I\cos A\\ \cos I\sin A\\ -\sin I\end{matrix} & \begin{matrix}-\sin I\sin A\\ \sin I\cos A\\ 0\end{matrix}\end{array}\right]\begin{bmatrix}0\\ 0\\ F_c\tan\Theta - \cot I\end{bmatrix} \\ &= \frac{\Delta D_k}{\|\boldsymbol{G}\|2\sqrt{2}}\begin{bmatrix} -F_sF_c\sin I\tan\Theta + F_s\cos I \\ F_c^2\sin I\tan\Theta - F_c\cos I \\ 0 \end{bmatrix} \end{aligned} \tag{24}$$

When $I \to 0$ with $G_x = 0$ and $\Delta G_y \to 0^+$, we can find the derivative of this fictive error source with respect to y axis accelerometer bias as the limit

$$\begin{aligned} \sqrt{\frac{1}{2}}d_{X_k} &= \lim_{\Delta G_y\to 0^+}\frac{\Delta D_k}{\|\boldsymbol{G}\|2\sqrt{2}}\begin{bmatrix} -\frac{(B_z(0+\Delta G_y^2)-G_z(0+\Delta G_yB_y))(0-\Delta G_yB_x)}{\|\boldsymbol{G}\|^3\|\boldsymbol{B}\|^2\cos^2\Theta\sin\Delta I}\tan\Theta + \frac{(0-\Delta G_yB_x)}{\|\boldsymbol{G}\|\|\boldsymbol{B}\|\cos\Theta\sin\Delta I}\cos\Delta I \\ \\ \frac{(B_z(0+\Delta G_y^2)-G_z(0+\Delta G_yB_y))^2}{\|\boldsymbol{G}\|^4\|\boldsymbol{B}\|^2\cos^2\Theta\sin\Delta I}\tan\Theta - \frac{(B_z(0+\Delta G_y^2)-G_z(0+\Delta G_yB_y))}{\|\boldsymbol{G}\|^2\|\boldsymbol{B}\|\cos\Theta\sin\Delta I}\cos\Delta I \\ \\ 0 \end{bmatrix} \\ &= \lim_{\Delta G_y\to 0^+}\frac{\Delta D_k}{\|\boldsymbol{G}\|2\sqrt{2}}\begin{bmatrix} \frac{(B_z\Delta G_y^2-G_z\Delta G_yB_y)\Delta G_yB_x}{\|\boldsymbol{G}\|^2\|\boldsymbol{B}\|^2\cos^2\Theta|\Delta G_y|}\tan\Theta - \frac{\Delta G_yB_x}{\|\boldsymbol{B}\|\cos\Theta|\Delta G_y|}\cos\Delta I \\ \\ \frac{(B_z\Delta G_y^2-G_z\Delta G_yB_y)^2}{\|\boldsymbol{G}\|^3\|\boldsymbol{B}\|^2\cos^2\Theta|\Delta G_y|}\tan\Theta - \frac{(B_z\Delta G_y^2-G_z\Delta G_yB_y)}{\|\boldsymbol{G}\|\|\boldsymbol{B}\|\cos\Theta|\Delta G_y|}\cos\Delta I \\ \\ 0 \end{bmatrix} \\ &= \frac{\Delta D_k}{2\sqrt{2}}\begin{bmatrix} \frac{-\sin\alpha_m}{\|\boldsymbol{G}\|} \\ \\ \frac{\cos\alpha_m}{\|\boldsymbol{G}\|} \\ \\ 0 \end{bmatrix} \end{aligned} \tag{25}$$

Note here that Equation 25 is not identical to Equation 22. However, analogously to Equation 22, it's limit does not exist, but the limit of its square does exist:

$$\frac{1}{2}d^2_{X_k} = \frac{\Delta D_k}{8}\begin{bmatrix} \frac{\sin^2\alpha_m}{\|\boldsymbol{G}\|^2} \\ \frac{\cos^2\alpha_m}{\|\boldsymbol{G}\|^2} \\ 0 \end{bmatrix} \tag{26}$$

Accelerometer x Axis Bias, Term 5

The 5^{th} fictive error source term corresponding to x accelerometer bias has a weighting function, for a stationary survey k, given by

$$\begin{aligned}
\sqrt{\frac{1}{2}}e_{X_k} &= \sqrt{\frac{1}{2}}\frac{\partial\Delta\boldsymbol{r}_k}{\partial\boldsymbol{p}_k}\frac{\partial\boldsymbol{p}_k}{\partial\boldsymbol{\epsilon}_i} \\
&= \frac{D_k - D_{k-1}}{2\sqrt{2}}\left[\begin{array}{c|c|c} \frac{1}{\Delta D_k}\frac{\partial\Delta\boldsymbol{r}_k}{\partial D_k} & \begin{matrix}\cos I\cos A \\ \cos I\sin A \\ -\sin I\end{matrix} & \begin{matrix}-\sin I\sin A \\ \sin I\cos A \\ 0\end{matrix}\end{array}\right]\begin{bmatrix} 0 \\ v_k^I \\ v_k^A \end{bmatrix} \\
&= \frac{\Delta D_k}{\|\boldsymbol{G}\|2\sqrt{2}}\left[\begin{array}{c|c|c} \frac{1}{\Delta D_k}\frac{\partial\Delta\boldsymbol{r}_k}{\partial D_k} & \begin{matrix}\cos I\cos A \\ \cos I\sin A \\ -\sin I\end{matrix} & \begin{matrix}-\sin I\sin A \\ \sin I\cos A \\ 0\end{matrix}\end{array}\right]\begin{bmatrix} 0 \\ 0 \\ -F_c\tan\Theta + \cot I \end{bmatrix} \\
&= \frac{\Delta D_k}{\|\boldsymbol{G}\|2\sqrt{2}}\begin{bmatrix} F_sF_c\sin I\tan\Theta - F_s\cos I \\ -F_c^2\sin I\tan\Theta + F_c\cos I \\ 0 \end{bmatrix}
\end{aligned} \tag{27}$$

When $I \to 0$ with $G_y = 0$ and $\Delta G_x \to 0^+$, we can find the derivative of this fictive error source with respect to x axis accelerometer bias as the limit

$$\sqrt{\frac{1}{2}}e_{X_k} = \lim_{\Delta G_x \to 0^+} \frac{\Delta D_k}{\|\boldsymbol{G}\|2\sqrt{2}} \begin{bmatrix} \frac{(\Delta G_x B_y - 0)(B_z(\Delta G_x^2 + 0) - G_z(\Delta G_x B_x + 0))}{\|\boldsymbol{G}\|^3\|\boldsymbol{B}\|^2 \cos^2\Theta \sin\Delta I} \tan\Theta - \frac{\cos\Delta I(\Delta G_x B_y - 0)}{\|\boldsymbol{G}\|\|\boldsymbol{B}\| \cos\Theta \sin\Delta I} \\ -\frac{(B_z(\Delta G_x^2 + 0) - G_z(\Delta G_x B_x + 0))^2}{\|\boldsymbol{G}\|^4\|\boldsymbol{B}\|^2 \cos^2\Theta \sin\Delta I} \tan\Theta + \frac{\cos\Delta I(B_z(\Delta G_x^2 + 0) - G_z(\Delta G_x B_x + 0))}{\|\boldsymbol{G}\|^2\|\boldsymbol{B}\| \cos\Theta \sin\Delta I} \\ 0 \end{bmatrix}$$

$$= \lim_{\Delta G_x \to 0^+} \frac{\Delta D_k}{\|\boldsymbol{G}\|2\sqrt{2}} \begin{bmatrix} \frac{\Delta G_x B_y(B_z \Delta G_x^2 - G_z \Delta G_x B_x)}{\|\boldsymbol{G}\|^2\|\boldsymbol{B}\|^2 \cos^2\Theta |\Delta G_x|} \tan\Theta - \frac{\cos\Delta I \Delta G_x B_y}{\|\boldsymbol{B}\| \cos\Theta |\Delta G_x|} \\ -\frac{(B_z \Delta G_x^2 - G_z \Delta G_x B_x)^2}{\|\boldsymbol{G}\|^3\|\boldsymbol{B}\|^2 \cos^2\Theta |\Delta G_x|} \tan\Theta + \frac{\cos\Delta I(B_z \Delta G_x^2 - G_z \Delta G_x B_x)}{\|\boldsymbol{G}\|\|\boldsymbol{B}\| \cos\Theta |\Delta G_x|} \\ 0 \end{bmatrix}$$

$$= \frac{\Delta D_k}{2\sqrt{2}} \begin{bmatrix} \frac{-\cos\alpha_m}{\|\boldsymbol{G}\|} \\ \frac{-\sin\alpha_m}{\|\boldsymbol{G}\|} \\ 0 \end{bmatrix} \tag{28}$$

So we have arrived at an answer using the limit as $\Delta G_x \to 0^+$. However, if we had used the limit as $\Delta G_x \to 0^-$ the sign of Equation 22 would be positive instead of negative. This means that there is no continuity in the limit, and therefore the limit does not exist. The weighting function for the 4^{th} fictive error term does not exist on its own. However, the limit of the square of the fictive error weighting term does exist, and that is what is used for error propagation:

$$\frac{1}{2}e_{X_k}^2 = \frac{\Delta D_k}{8} \begin{bmatrix} \frac{\cos^2\alpha_m}{\|\boldsymbol{G}\|^2} \\ \frac{\sin^2\alpha_m}{\|\boldsymbol{G}\|^2} \\ 0 \end{bmatrix} \tag{29}$$

Accelerometer y Axis Bias, Term 5

These fictive error sources corresponding to y accelerometer bias have weighting functions, for a stationary survey k, given by

$$\begin{aligned}
\sqrt{\frac{1}{2}} e_{X_k} &= \sqrt{\frac{1}{2}} \frac{\partial \Delta \boldsymbol{r}_k}{\partial \boldsymbol{p}_k} \frac{\partial \boldsymbol{p}_k}{\partial \boldsymbol{\epsilon}_i} \\
&= \frac{D_k - D_{k-1}}{2\sqrt{2}} \left[\frac{1}{\Delta D_k} \frac{\partial \Delta \boldsymbol{r}_k}{\partial D_k} \middle| \begin{array}{c} \cos I \cos A \\ \cos I \sin A \\ -\sin I \end{array} \middle| \begin{array}{c} -\sin I \sin A \\ \sin I \cos A \\ 0 \end{array} \right] \begin{bmatrix} 0 \\ v_k^I \\ v_k^A \end{bmatrix} \\
&= \frac{\Delta D_k}{\|\boldsymbol{G}\| 2\sqrt{2}} \left[\frac{1}{\Delta D_k} \frac{\partial \Delta \boldsymbol{r}_k}{\partial D_k} \middle| \begin{array}{c} \cos I \cos A \\ \cos I \sin A \\ -\sin I \end{array} \middle| \begin{array}{c} -\sin I \sin A \\ \sin I \cos A \\ 0 \end{array} \right] \begin{bmatrix} 0 \\ -\cos I \\ F_s \cos I \tan \Theta \end{bmatrix} \\
&= \frac{\Delta D_k}{\|\boldsymbol{G}\| 2\sqrt{2}} \begin{bmatrix} -F_c \cos^2(I) - F_s^2 \sin I \cos I \tan \Theta \\ -F_s \cos^2(I) + F_s F_c \sin I \cos I \tan \Theta \\ \sin I \cos I \end{bmatrix}
\end{aligned} \tag{30}$$

When $I \to 0$ with $G_x = 0$ and $\Delta G_y \to 0^+$, we can find the derivative of this fictive error source with respect to y axis accelerometer bias as the limit

$$\begin{aligned}
\sqrt{\frac{1}{2}} e_{X_k} &= \lim_{\Delta G_x \to 0^+} \frac{\Delta D_k \cos \Delta I}{\|\boldsymbol{G}\| 2\sqrt{2}} \begin{bmatrix} -\frac{\cos \Delta I (B_z(0+\Delta G_y^2) - G_z(0+\Delta G_y B_y))}{\|\boldsymbol{G}\|^2 \|\boldsymbol{B}\| \cos \Theta \sin \Delta I} - \frac{(0-\Delta G_y B_x)^2}{\|\boldsymbol{G}\|^2 \|\boldsymbol{B}\|^2 \cos^2 \Theta \sin \Delta I} \tan \Theta \\ -\frac{\cos \Delta I (0-\Delta G_y B_x)}{\|\boldsymbol{G}\| \|\boldsymbol{B}\| \cos \Theta \sin \Delta I} + \frac{(B_z(0+\Delta G_y^2) - G_z(0+\Delta G_y B_y))(0-\Delta G_y B_x)}{\|\boldsymbol{G}\|^3 \|\boldsymbol{B}\|^2 \cos^2 \Theta \sin \Delta I} \tan \Theta \\ \sin \Delta I \end{bmatrix} \\
&= \lim_{\Delta G_x \to 0^+} \frac{\Delta D_k \cos \Delta I}{\|\boldsymbol{G}\| 2\sqrt{2}} \begin{bmatrix} -\frac{\cos \Delta I (B_z \Delta G_y^2 - G_z \Delta G_y B_y)}{\|\boldsymbol{G}\| \|\boldsymbol{B}\| \cos \Theta |\Delta G_y|} - \frac{(\Delta G_y B_x)^2}{\|\boldsymbol{G}\| \|\boldsymbol{B}\|^2 \cos^2 \Theta |\Delta G_y|} \tan \Theta \\ \frac{\cos \Delta I \Delta G_y B_x}{\|\boldsymbol{B}\| \cos \Theta |\Delta G_y|} - \frac{(B_z \Delta G_y^2 - G_z \Delta G_y B_y) \Delta G_y B_x}{\|\boldsymbol{G}\|^2 \|\boldsymbol{B}\|^2 \cos^2 \Theta |\Delta G_y|} \tan \Theta \\ \sin \Delta I \end{bmatrix} \\
&= \frac{\Delta D_k}{2\sqrt{2}} \begin{bmatrix} \frac{\cos \alpha_m}{\|\boldsymbol{G}\|} \\ \frac{\sin \alpha_m}{\|\boldsymbol{G}\|} \\ 0 \end{bmatrix}
\end{aligned} \tag{31}$$

Note here that Equation 31 is not identical to Equation 28. However, analogously to Equation

28, it's limit does not exist, but the limit of its square does exist:

$$\frac{1}{2}e^2_{X_k} = \frac{\Delta D_k}{8}\begin{bmatrix} \frac{\cos^2\alpha_m}{\|\boldsymbol{G}\|^2} \\ \frac{\sin^2\alpha_m}{\|\boldsymbol{G}\|^2} \\ 0 \end{bmatrix} \tag{32}$$

Accumulating of Position Uncertainty Due to Accelerometer x/y Axis Bias Terms 4 and 5

Recall that weighting functions of the fictive error terms are defined, for example, such that

$$\sqrt{\frac{1}{2}}d_{X_j} = \sqrt{\frac{1}{2}}\frac{\partial \Delta \boldsymbol{r}_j}{\partial \boldsymbol{p}_j}\frac{\partial \boldsymbol{p}_j}{\partial \boldsymbol{\epsilon}_i} \tag{33}$$

Thus, if we wanted to determine the position uncertainty at survey station k due to one such systematic error source, we would use

$$E\left[\Delta\tilde{\boldsymbol{r}}_k\Delta\tilde{\boldsymbol{r}}_k^T\right] = E\left[\sum_{j=2}^{k}\left(\frac{\partial \Delta \boldsymbol{r}_j}{\partial \boldsymbol{p}_j}\frac{\partial \boldsymbol{p}_j}{\partial \boldsymbol{\epsilon}_i} + \frac{\partial \Delta \boldsymbol{r}_j}{\partial \boldsymbol{p}_{j-1}}\frac{\partial \boldsymbol{p}_{j-1}}{\partial \boldsymbol{\epsilon}_i}\right)\epsilon_i^2\sum_{m=2}^{k}\left(\frac{\partial \Delta \boldsymbol{r}_m}{\partial \boldsymbol{p}_m}\frac{\partial \boldsymbol{p}_m}{\partial \boldsymbol{\epsilon}_i} + \frac{\partial \Delta \boldsymbol{r}_m}{\partial \boldsymbol{p}_{m-1}}\frac{\partial \boldsymbol{p}_{m-1}}{\partial \boldsymbol{\epsilon}_i}\right)^T\right] \tag{34}$$

Using the notation

$$\begin{aligned}\sqrt{\frac{1}{2}}d_{X_{j,j-1}} &= \sqrt{\frac{1}{2}}\frac{\partial \Delta \boldsymbol{r}_j}{\partial \boldsymbol{p}_{j-1}}\frac{\partial \boldsymbol{p}_{j-1}}{\partial \boldsymbol{\epsilon}_i} \\ \sqrt{\frac{1}{2}}d_{X_{j,j}} &= \sqrt{\frac{1}{2}}\frac{\partial \Delta \boldsymbol{r}_j}{\partial \boldsymbol{p}_j}\frac{\partial \boldsymbol{p}_j}{\partial \boldsymbol{\epsilon}_i}\end{aligned} \tag{35}$$

Equation 34 can be rewritten

$$\begin{aligned}E\left[\Delta\tilde{\boldsymbol{r}}_k\Delta\tilde{\boldsymbol{r}}_k^T\right] &= \frac{\sigma_i^2}{2}\sum_{j=2}^{k}\sum_{m=2}^{k}\left\{d_{X_{j,j}}d_{X_{m,m}}\rho(\tau_j,\tau_m) + d_{X_{j,j}}d_{X_{m,m-1}}\rho(\tau_j,\tau_{m-1})\right. \\ &\left. + d_{X_{j,j-1}}d_{X_{m,m}}\rho(\tau_{j-1},\tau_m) + d_{X_{j,j-1}}d_{X_{m,m-1}}\rho(\tau_{j-1},\tau_{m-1})\right\}\end{aligned} \tag{36}$$

where it has been assumed that the variance of each bias term individually is σ_i^2. Because all of the d_X values for a single fictive error source have the same dependence on the sign of that error source in terms of their limit as $I \to 0$, this dependence will cancel out in the covariance calculation, meaning the limit will exist. Furthermore, for fictive error term 4, for any two survey stations terms of the form $d_{X_{j,j}}d_{X_{m,m}}\rho(\tau_j,\tau_m)$ will only be non-zero if the correlation term, $\rho(\tau_j,\tau_m)$, is equal to 1, meaning $\tau_j = \tau_m$. Thus, the only non-zero terms in Equation 36 will be of the form

$$\frac{1}{2}d_{X_{j,j}}d^T_{X_{m,m}} = \frac{\Delta D_j \Delta D_m}{8\|\boldsymbol{G}\|^2}\begin{bmatrix} \sin\alpha_{m_j}\sin\alpha_{m_j} & -\sin\alpha_{m_j}\cos\alpha_{m_j} & 0 \\ -\cos\alpha_{m_j}\sin\alpha_{m_j} & \cos\alpha_{m_j}\cos\alpha_{m_j} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{37}$$

This holds for both the contributions from the x and y accelerometer biases. Therefore, it is valid to combine the effects of fictive error term 4 from the x and y biases to get the combined accelerometer bias contribution from fictive error term 4:

$$\begin{aligned} E\left[\Delta\tilde{\boldsymbol{r}}_k\Delta\tilde{\boldsymbol{r}}_k^T\right] &= \sigma_i^2\sum_{j=2}^{k}\sum_{m=2}^{k}\left\{d_{X_{j,j}}d_{X_{m,m}}\rho(\tau_j,\tau_m)+d_{X_{j,j}}d_{X_{m,m-1}}\rho(\tau_j,\tau_{m-1})\right.\\ &\left.+d_{X_{j,j-1}}d_{X_{m,m}}\rho(\tau_{j-1},\tau_m)+d_{X_{j,j-1}}d_{X_{m,m-1}}\rho(\tau_{j-1},\tau_{m-1})\right\} \end{aligned} \quad (38)$$

This term is referred to in the ISCWSA error model paradigm as ABXY-TI1.

Similarly for Term 5, the analog of Equation 37 for any two survey stations is

$$\frac{1}{2}e_{X_{j,j}}e^T_{X_{m,m}} = \frac{\Delta D_j\Delta D_m}{8\|\boldsymbol{G}\|^2}\begin{bmatrix}\cos\alpha_{m_j}\cos\alpha_{m_j} & \cos\alpha_{m_j}\sin\alpha_{m_j} & 0\\ \sin\alpha_{m_j}\cos\alpha_{m_j} & \sin\alpha_{m_j}\sin\alpha_{m_j} & 0\\ 0 & 0 & 0\end{bmatrix} \quad (39)$$

which holds for both the x and y bias fictive error terms. Thus, it is valid to combine the effects of fictive error term 5 from the x and y biases to get the combined accelerometer bias contribution from fictive error term 5:

$$\begin{aligned} E\left[\Delta\tilde{\boldsymbol{r}}_k\Delta\tilde{\boldsymbol{r}}_k^T\right] &= \sigma_i^2\sum_{j=2}^{k}\sum_{m=2}^{k}\left\{e_{X_{j,j}}e_{X_{m,m}}\rho(\tau_j,\tau_m)+e_{X_{j,j}}e_{X_{m,m-1}}\rho(\tau_j,\tau_{m-1})\right.\\ &\left.+e_{X_{j,j-1}}e_{X_{m,m}}\rho(\tau_{j-1},\tau_m)+e_{X_{j,j-1}}e_{X_{m,m-1}}\rho(\tau_{j-1},\tau_{m-1})\right\} \end{aligned} \quad (40)$$

where it has been assumed that the variance of each bias term individually is σ_i^2. This term is referred to in the ISCWSA error model paradigm as ABXY-TI2.

Analysis of Equations 37 and 39 shows that, for each entry in Equations 38 and 40 the off-diagonal terms will drop out, and the addition of the on-diagonal terms will result in an identity matrix. This is equivalent to using the vertical well weighting functions:

$$\boxed{\text{ABXY-TI1 } \textit{weighting function} = \frac{\Delta D}{2}\begin{bmatrix}\frac{1}{\|\boldsymbol{G}\|}\\ 0\\ 0\end{bmatrix}} \quad (41)$$

and

$$\boxed{\text{ABXY-TI2 } \textit{weighting function} = \frac{\Delta D}{2}\begin{bmatrix}0\\ \frac{1}{\|\boldsymbol{G}\|}\\ 0\end{bmatrix}} \quad (42)$$

Note that the term weighting function here refers to the partials of the error source with respect to the position vector. These weighting functions are appealing for use in toolface-independent error models because they have no toolface dependence of any kind and contain no terms that are not defined, such as azimuth.

Comparison with Copsegrove Memo Formulation

The Copsegrove memo CDR-SM-03, Rev. 4 uses a different formulation of the accelerometer x and y bias functions in a vertical hole. The ABXY-TI1 term is handled in the standard, non-singular fashion. The net result is

$$\frac{\partial \Delta \boldsymbol{r}_k}{\partial \boldsymbol{p}_k}\frac{\partial \boldsymbol{p}_k}{\partial \boldsymbol{\epsilon}_i} = \frac{D_k - D_{k-1}}{2}\left[\begin{array}{c|c|c} \frac{1}{\Delta D_k}\frac{\partial \Delta \boldsymbol{r}_k}{\partial D_k} & \begin{array}{c}\cos I \cos A \\ \cos I \sin A \\ -\sin I\end{array} & \begin{array}{c}-\sin I \sin A \\ \sin I \cos A \\ 0\end{array}\end{array}\right]\left[\begin{array}{c} 0 \\ -\frac{\cos I}{\|\boldsymbol{G}\|} \\ \frac{\tan \Theta \cos I \sin A}{\|\boldsymbol{G}\|}\end{array}\right] \tag{43}$$

If we assume that when $I = 0$, the computer code calculates $A = 0$ and $\alpha = 0$, then this weighting function reduces to

$$\frac{\partial \Delta \boldsymbol{r}_k}{\partial \boldsymbol{p}_k}\frac{\partial \boldsymbol{p}_k}{\partial \boldsymbol{\epsilon}_i} = \frac{\Delta D_k}{2}\left[\begin{array}{c} -\frac{1}{\|\boldsymbol{G}\|} \\ 0 \\ 0\end{array}\right] \tag{44}$$

which, except for the sign, is equivalent to Equation 41.

The vertical well weighting function for the ABXY-TI2 term is given as

$$\frac{\partial \Delta \boldsymbol{r}_k}{\partial \boldsymbol{p}_k}\frac{\partial \boldsymbol{p}_k}{\partial \boldsymbol{\epsilon}_i} = \left[\begin{array}{c} -\frac{\sin A}{\|\boldsymbol{G}\|} \\ \frac{\cos A}{\|\boldsymbol{G}\|} \\ 0\end{array}\right] \tag{45}$$

Under the same conditions as above, for $I = 0$ this reduces to

$$\frac{\partial \Delta \boldsymbol{r}_k}{\partial \boldsymbol{p}_k}\frac{\partial \boldsymbol{p}_k}{\partial \boldsymbol{\epsilon}_i} = \left[\begin{array}{c} 0 \\ \frac{1}{\|\boldsymbol{G}\|} \\ 0\end{array}\right] \tag{46}$$

which, except for multiplier $\frac{\Delta D_k}{2}$, is equivalent to Equation 42. Clearly the $\frac{\Delta D_k}{2}$ multiplier should be included (simulation confirms this), and this was likely the intention of the memo. Still, care must be taken to implement the ABXY-TI2 weighting function correctly, given the incomplete presentation in CDR-SM-03, Rev. 4. Leaving the multiplier off can effectively remove all contribution of accelerometer bias to position uncertainty in the East direction during the vertical portion of a well, distorting the error ellipse. Otherwise, the two formulations agree perfectly, as can be seen in Figure 2 for the rotating case.

References

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[Williamson(2000)] H. S. Williamson. Accuracy prediction for directional measurement while drilling, SPE 67616. *SPE Drilling and Completion*, 15(4), December 2000.

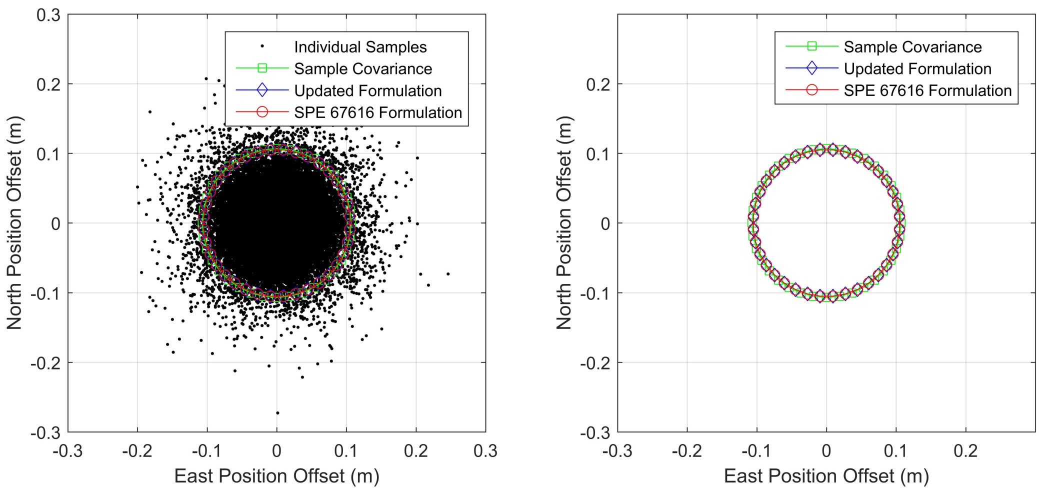

Figure 2: Comparison of Horizontal Position Uncertainty Predictions in a Vertical Well Under Rotating Conditions After 300 Meters. Plots are Identical Except the Left Plot Shows the Samples Drawn from a 50,000 Run Monte Carlo Analysis.